An Enquiry into the different MODES *of* DEMONSTRATION, *by which the* VELOCITY *of* SPOUTING FLUIDS *has been inveſtigated* a priori. *By the Rev.* M. YOUNG, *D. D. F. T. C. D. and M. R. I. A.*

THE antients, as Doctor Jurin informs us, had no knowledge of any meaſure of the flux of water, except that fallacious and uncertain meaſure derived from a perpendicular ſection of the ſtream alone, without any regard to the velocity with which it flows. Benedict Caſtelli, an Italian, and friend of Gallileo, was the firſt who opened the way to a true meaſure. The neceſſity of guarding continually againſt the damages from the overflowings of the rivers in Italy, induced Urban the Eighth, who had invited him to Rome as a teacher of mathematics, to requeſt he would apply himſelf to this ſubject. The reſult of his enquiries is contained in his treatiſe entitled *Della meſura dell' acque correnti*; which meaſure he found to depend on the area of the ſection and the velocity of the water conjointly. The fundamental principle of this and other queſtions in hydraulics is the determination of the actual velocity with which water ſpouts

Read March 1, 1788.

 from

from an aperture in the bottom or ſide of a veſſel; but there is none which ſeems to have produced greater perplexity.

SCARCELY can one writer be found who acquieſces in the ſolution of another. Even the great Newton, who paid particular attention to this ſubject, is not very conſiſtent with himſelf. In the firſt edition of his Principia he endeavours to prove, that the velocity of the ſpouting water is equal to that which a heavy body would acquire in falling through half the height of the water above the aperture; in his ſecond and third editions he relinquiſhes this calculation, and demonſtrates that the velocity is that which would be acquired in falling through the entire altitude. Yet he immediately ſubjoins an account of experiments which he made with a view to aſcertain this point, and which ſeem inconſiſtent with the demonſtration he adheres to, though very conſonant to that which he rejects.

THE demonſtration which he gives in the firſt edition appears at firſt ſight to be unexceptionable, and has accordingly been received by the learned Emerſon, Whiſton, Mr. Wildbore in Hutton's *Miſcellanea Mathem.* and other good philoſophers. It is to this effect:

IF a veſſel be filled with water, and perforated in the bottom ſo as that the water may flow through the aperture, it is manifeſt that the bottom will ſuſtain the weight of all the water except the weight of that part perpendicularly incumbent over the orifice. For if the orifice be cloſed by any obſtacle, that obſtacle will ſuſtain the weight of the water perpendicularly incumbent on it, and the bottom

bottom of the veſſel will ſuſtain the reſt. But on removing the obſtacle, the bottom of the veſſel will be preſſed in the ſame manner as before, and the weight which was ſuſtained by the obſtacle being now no longer ſupported, will produce an efflux of the water through the orifice. Whence it follows, that the motion of the whole effluent water is ſuch as can be produced by the weight of the column of water incumbent over the orifice. For every particle of water deſcends by its own weight, as far as it is not impeded, with a uniformly accelerated motion, and as far as it is impeded it will preſs the obſtacle; that obſtacle is either the bottom of the veſſel or the inferior deſcending water, and therefore that part of the weight which the bottom of the veſſel does not ſuſtain will preſs the effluent water, and generate a motion proportional to it.

Let F denote the area of the orifice, A the altitude of the column of water over the orifice, V the velocity which a heavy body would acquire in falling through the height A in the time T, and x the velocity of the effluent water. Since in the time T, a ſpace equal to 2 A would be deſcribed with the velocity V, a ſpace equal to $\frac{2Ax}{V}$ will be the ſpace deſcribed in the ſame time with the velocity x. This, therefore, will be the length of the column diſcharged in the time T, and the magnitude of this cylinder will be $\frac{2AxF}{V}$, and its quantity of motion $=\frac{2Ax^2F}{V}$. But the quantity of motion, which, in the ſame time, would be generated in the column of water incumbent on the orifice, if it were to fall freely as an heavy body through a ſpace equal to

its altitude, would be A F V. And theſe quantities of motion are equal, being both generated in equal times, by the ſame generating force; that is, $AFV = \frac{2Ax^2F}{V}$, whence $V^2 = 2x^2$, and $V : x :: \sqrt{2 : 1}$. Conſequently ſince V is the velocity which a heavy body would acquire in falling through the entire altitude A, one half of A will be the ſpace through which a body deſcending will acquire that velocity x with which the water flows from the orifice.

The latent fallacy of this argument conſiſts in this, that each plate of water is ſuppoſed to be ſucceſſively diſcharged with a uniform velocity, and the quantity of motion generated in every little portion of time in which each plate is diſcharged is meaſured by the plate drawn into the uniform velocity of the efflux. But this, on a little conſideration, will be found not to be a true ſtatement of the caſe; for every plate of water is diſcharged in time, and its velocity is uniformly encreaſed from nothing, during the deſcent of the plate through its own altitude, at the end of which little portion of time it attains that ultimate velocity with which it afterwards continues to move uniformly. Hence, therefore, it follows that the quantity of motion really generated during the time of the diſcharge of each plate of water is but half that which is determined by ſuppoſing the water to be diſcharged at once with its full velocity. The correction of this error will lead us to a true ſolution of the queſtion. Let the time of a body's fall through the height A be divided into an indefinite number of little portions, each equal to the time in which a plate of water is diſcharged by deſcending through its own

own altitude. The quantity of motion generated in the cylinder in the time of the fall is equal to the ſum of the quantities of motion generated in that cylinder in all the little portions of time into which the whole time is divided, taken ſeparately; that is, equal to the ſum of the quantities of motion generated by the preſſure of that cylinder, in the ſame little portions of time, in the ſucceſſive plates of water (becauſe equal forces generate equal quantities of motion in the ſame time); that is, equal to half the quantity of motion in theſe plates as meaſured by the ultimate velocity continued uniform for theſe portions of time. Whence it follows, that the quantity of motion generated in the incumbent cylinder, if it were ſuppoſed to fall freely through its height, is equal to half the quantity of motion of the cylinder ſuppoſed to be diſcharged in the ſame time with a uniform velocity. That is, $AFV = \frac{Ax^2F}{V}$, whence $V^2 = x^2$, and $V = x$; that is, the ultimate velocity with which the water is diſcharged is equal to that which a heavy body would acquire in falling through the entire height of the water above the orifice.

The mode of demonſtration which Sir Iſaac purſues in the ſecond and third editions of his Principia, and which has been admitted and diſcuſſed at large by Jurin, Maclaurin, Robinſon, and other mathematicians, is this: Let MNCD be a cylindric veſſel filled with water to the height AB; CD its bottom parallel to the horizon; EF a circular hole in the bottom, and IG a perpendicular to the horizon paſſing through the centre of the hole. Newton then ſuppoſes water to be poured in at the upper ſurface AB as faſt as it ſubſides by the efflux of the water through the aperture

aperture E F; and ſince the water is continually accelerated from the upper ſurface to the bottom, where it is diſcharged, by the action of gravity, the body of deſcending water will contract in breadth according as the velocity encreaſes, ſo as to move in a regular curve A K E, which he calls the cataract of deſcending water. And becauſe the water, in its deſcent, ſuffers no other reſiſtance than what ariſes from the friction or mutual adheſion of the particles, which in the preſent caſe is ſuppoſed evaneſcent, it follows, that the particles will deſcend to the hole with a velocity uniformly accelerated, and conſequently that their velocity in the aperture will be the ſame as if they had deſcended in the vertical line H G. Now becauſe at the very inſtant that the water flows from the aperture the ſurface A B ſubſides, and the water is ſupplied as faſt at that ſurface as it iſſues from the orifice, it follows, that by diſcovering the velocity with which the water is poured in at A B to ſupply the waſte, the velocity with which it iſſues from the orifice will be alſo aſcertained. For ſuppoſe I H to be the height from which a body muſt fall in order to acquire the velocity with which the water is poured in, ſince it is uniformly accelerated from thence to the orifice by the action of gravity, it follows, that the velocity of the effluent water will be that which a heavy body would acquire in falling down the height I G.

He then proceeds to calculate the height I G; and if S denote the ſurface of the water at A B, A the orifice, and H the height of the water, he ſhews that I G will be equal to the quantity $HG \times \frac{S^2}{S^2 - A^2}$.

But

But many reaſons concur to render us ſuſpicious of the truth of this reaſoning: In the firſt place, it is extremely improbable that the water ſhould deſcend in this regular cataract, leaving the fluid in the ambient ſpace at reſt; and it appears to be falſe in fact, by obſerving the motion of light particles ſuſpended in the water, whoſe motion does not appear to be confined within the bounds of the cataract, or to be performed in that regular curve which the reaſoning requires. Secondly, Newton ſuppoſes that the water which iſſues with this velocity deſcends from the upper ſurface; if this were ſo, the ſpouting fluid could not attain its full velocity till a cylinder of it had been diſcharged, whoſe baſe is equal to the area of the orifice, and height equal to that of the fluid; but this is not the caſe, for the loweſt plate or ſmalleſt quantity of the fluid will be diſcharged with its full velocity. Thirdly, ſince the orifice is leſs than the upper ſurface of the water, it would follow that the altitude I G would be greater than H G; that is, the velocity of the ſpouting fluid would be greater than that which a heavy body would acquire in falling through the height of the veſſel; and that exceſs would be greater the larger the aperture; ſo that by encreaſing the aperture we might encreaſe the velocity of the ſpouting fluid at pleaſure. But this appears not by any means to be true in fact; for we can never produce, by any variation of the orifice, a velocity greater than that which a heavy body would acquire in falling through the height of the fluid.

Doctor Helsham's demonſtration of this propoſition is to the following effect: If we ſuppoſe the column of water which ſtands directly over the orifice to be divided into an indefinite number of plates of an equal, but exceedingly ſmall thickneſs, we

we muſt allow that whatever be the force of gravity wherewith the uppermoſt plate preſſes upon the ſecond, the ſecond preſſes on the third with a double force, and the third upon the fourth with a triple force, and ſo on; ſo that the plate which is next the orifice is preſſed downward by the joint gravities of the ſeveral plates which lie above it, and likewiſe by the force of its own gravity, inaſmuch as there is no other plate beneath it whereon to reſt; conſequently from its own gravity, and that of the ſeveral plates above it, it does all at once receive as many equal impreſſions from gravity, as it would ſucceſſively in falling down the height of the water; and of courſe muſt paſs through the orifice with the ſame velocity that it would acquire in falling down that height.

THIS demonſtration appears to be defective in this reſpect, that it does not take into account the time in which the force accelerating the diſcharge of the water acts; for it is evident, that the greater the velocity with which the loweſt plate of water is diſcharged through the orifice the ſhorter will be the time during which it is accelerated by the preſſure of the incumbent fluid. By neglecting this circumſtance, it would follow, from Doctor Helſham's reaſoning, that the velocity ſhould be in the direct ſimple, not ſubduplicate ratio of the height of the fluid, the velocity generated being, *ceteris paribus*, as the accelerating force, that is, as the height of the column of water ſtanding directly above the orifice. If, indeed, this time be taken into conſideration, the inference will be legitimate. Thus the velocity generated in the iſſuing plate of water will be as the accelerating force and the time of its action conjointly, the plate, that is, the quantity of matter moved, being given; but the time

in

in which the accelerating force acts on the plate, is inverfely as the velocity with which the plate iffues; therefore the fquare of the velocity is directly as the force, and the velocity as the fquare root of the force, that is, as the fquare root of the height of the water above the orifice. But the actual velocity of the effluent water would not even thus be afcertained.

The Abbè Winkler's demonftration is built on the fame foundation with Helfham's.

Musschenbroeck's demonftration of this principle is liable to a three-fold objection: Firft, it is founded on a falfe meafure of the force of bodies in motion, to wit, the quantity of matter and the fquare of the velocity. Secondly, it involves a confufion of what is an equal ratio with a ratio of equality. Thirdly, it implies that equal forces generate equal velocities, without any regard to the times in which they act, or the quantities of matter which they move.

Varignon proves only, that the velocities of fpouting fluids are in the fubduplicate ratios of the heights of the fluids above the apertures, but does not afcertain the actual velocity, which is the principal object of enquiry. See Acad. Science. An. 1703.

Belidor's demonftration is fubject to the fecond imperfection of Muffchenbrœck's. For from proving, after Varignon, that the velocity of the effluent water is proportional to the fquare root of the height of the water, and therefore follows the fame law of acceleration with that of falling bodies, he concludes, that the velocity of the fpouting water is actually the fame which a

heavy body would acquire in falling through the height of the fluid.

PROFESSOR Gravefende, who has confidered this fubject with particular attention, has alfo given us a demonftration, that the velocity of the effluent water is equal to that which a body would acquire in falling through the entire height of the fluid. But it appears liable to the following objections: Firft, it fuppofes, that the velocities communicated to equal quantities of matter, in moving through equal fpaces, are directly as the generating forces, without any regard to the time in which thefe fpaces are run over by the bodies moved. And fecondly, it fuppofes that the forces acquired by the falling bodies are equal when the heights are inverfely as the maffes; whereas they are equal only when the maffes are in the inverfe fubduplicate of the heights.

I HAVE already fhewn how the demonftration given in the firft edition of the Principia, when duly corrected, affords a legitimate folution of this problem; and the fame conclufion may, I think, be thus otherwife made out in an unexceptionable manner.

LET MNOP reprefent a veffel of water filled to the level GH; MP the bottom, in which is the aperture CD; CIKD the column of water ftanding directly above the orifice, and CABD the loweft plate of water immediately contiguous to the aperture. Alfo let v denote the velocity which a heavy body would acquire in falling freely through the height BD of the plate, and x the velocity acquired by the fame plate during its defcent through the fame fpace until it is difcharged by the preffure of the column CIKD.

SUPPOSE

Suppose the loweſt plate of water ACBD to fall as a heavy body through the height BD, its moving force will be its own weight. Again, ſuppoſe it to be accelerated by its own weight and that of the incumbent water, that is, by the weight of the column CIKD through the ſame ſpace, that is, while it is accelerated from quieſcence until it is actually diſcharged. The velocity in the former caſe will be to that in the latter as the moving forces and the times in which they act directly, and the quantities of matter moved inverſely. But the moving forces are to each other as the heights BD and KD; the times in which they act are inverſely as the velocities, the ſpace through which the body is accelerated being given; and the quantities of matter moved are equal; therefore $v : x :: \frac{BD}{v} : \frac{KD}{x}$; conſequently $v^2 : x^2 :: BD : KD$. But v is the velocity which a heavy body would acquire in falling through the ſpace BD; therefore x, the velocity of the ſpouting fluid, is that which a heavy body would acquire in falling through KD, the height of the fluid above the orifice.

In the ſame manner it may be ſhewn, that if a pipe be inſerted horizontally in the veſſel NOMP, the plate of water ACBD will be diſcharged with the ſame velocity as before, whatever be the thickneſs of the plate; this velocity not depending on a continual acceleration through the length of the tube, otherwiſe the effluent water could not attain its full velocity, until a column had been diſcharged whoſe baſe is equal to the orifice and height equal to the length of the tube: whereas we find by experience, that this full velocity can be attained by the thinneſt plate which we can let eſcape from the aperture.

WHAT is here ſaid of the velocity of the effluent water is true only of the middle filament of particles which iſſue through the centre of the aperture, and which ſuffer no other retardation than what ariſes from the reſiſtance of the air, and their mutual adheſion and attrition againſt each other. But thoſe which iſſue near the edges of the aperture undergo a greater attrition, and therefore ſuffer a greater retardation. Hence it follows, that the mean velocity of the whole column of effluent water will be conſiderably leſs than according to theory.

SIR Iſaac Newton, who examined every ſubject that came before him with peculiar accuracy, firſt diſcovered a contraction in the vein of effluent water; and he found, that at the diſtance of about a diameter of the orifice, the ſection of the vein contracted nearly in the ſubduplicate ratio of 2 to 1. Hence he concluded that the velocity of the water, after its exit from the aperture, was encreaſed in this proportion, the ſame quantity paſſing in the ſame time through a narrower ſpace. Now, from the quantity of water diſcharged in a given time through that narrow ſection, he found that its velocity there was that which a heavy body would acquire in falling through the height of the water above the orifice; and ſince the velocity there was greater than immediately in the orifice in the ſubduplicate ratio of 2 to 1, he concluded that the velocity of the effluent water in the orifice was equal to that which a heavy body would acquire in falling through half the altitude. But all this is true only of the mean velocity; for there is no cauſe which can actually accelerate the water after its exit from the orifice, whatever cauſes may contribute to its retardation. The manner in which the mean velocity

city of the water is encreaſed after its diſcharge, though the actual velocity of the ſeveral particles continues unvaried, may be thus explained: the particles which iſſue near the ſides of the orifice proceed converging towards the axis of the vein, and with a retarded motion, upon account of their attrition againſt the ſides of the orifice; and as the central particles move faſter than thoſe which are farther from the axis, each plate of water after leaving the orifice, will aſſume a curved form, the concavity of which will reſpect the orifice. Let E F be the diameter of the vein where narroweſt, and A B the diameter of the orifice; the line of particles A B, which leave the orifice at the ſame inſtant, will aſſume à curvilineal poſition E G F, the central particles at G moving faſter than the extreme ones at E and F; the particles, therefore, in the diameter of the vein between E and F are ſupplied from the plates of water which iſſued ſucceſſively after EGF; and theſe extreme particles being thus diminiſhed in number, the central particles continuing nearly the ſame, the mean velocity muſt be encreaſed, becauſe that velocity is found by dividing the ſum of the velocities of all the particles by their number, and the number of particles which move with the greateſt velocity bears a greater proportion to the whole number in the narrow ſection of the vein at E F than in the orifice. In ſhort, to expreſs myſelf perhaps more clearly, the particles in the diameter A B, without being accelerated after their exit from the orifice, paſs through a leſs ſpace, becauſe they arrive at that ſpace in different times. It appears, therefore, that the actual velocity of the effluent water is not encreaſed after its diſcharge from the orifice, the contraction of the vein not inferring any ſuch augmentation, and there being no cauſe by which it could be pro-

duced.

duced. That the velocity with which the water is difcharged is really fuch as the theory gives, is fufficiently confirmed by the well-known experiment that water fpouts to the level of the refervoir, except fo far as it is impeded by external caufes. But though the velocity with which water, unrefifted in its paffage, iffues through the aperture may be thus afcertained by the height or diftance to which it fpouts, yet the mean velocity of the whole body of effluent water, taking in all caufes obftructing its difcharge which feem to lie beyond the reach of calculation, will be confiderably lefs than this, and can be eftimated in general by the analogy of experiment only.

The manner of making this eftimate is to find by experiment the quantity difcharged, in a given time, in any particular cafe, and reducing it to a column whofe bafe is equal to the aperture, the height of that column will be the fpace which would be defcribed in the propofed time by all the particles moving with a common velocity. The height of a column difcharged in any number of feconds t is equal to $2\,t\,F\sqrt{l\,D}$, F denoting the area of the aperture in fquare inches, l the fpace which a body defcribes in one fecond, falling freely from a ftate of reft, and D the height from which a heavy body muft fall in order to acquire the velocity of the effluent fluid. A cubic inch of water weighs ,52746 parts of a troy ounce; therefore $\left(\frac{W}{14{,}65\,t\,F}\right)^2 = D$, the height from which a body muft fall to acquire the mean velocity with which the water fpouts out, W denoting the weight of the water in troy ounces. Thus fuppofe a cylinder 20 inches high, kept conftantly filled with water, is found to difcharge 20 ounces troy through a circular aperture of

of $\frac{1}{5}$ of an inch diameter in 15 feconds; by the foregoing formula the height from which a body muft fall to acquire the velocity with which the water is difcharged will be 8,35 inches; that, is a fpace which is to the whole height of the water above the orifice as 10 to 24 nearly. But as it is a difficult matter to keep the fluid always at the fame height, without encreafing the preffure by pouring it in, it may perhaps be confidered a more exact method to calculate *a priori* the time in which the veffel ought to difcharge itfelf, and noting the actual time of the difcharge by experiment, to diminifh the velocity of the efflux determined according to theory in the fame ratio in which the time of the difcharge has been encreafed.

Now as the bafe of the veffel is to the orifice, fo is the time in which the veffel would empty itfelf to that in which a body would fall freely through the height of the water in the veffel: let therefore B denote the bafe of a cylindrical or prifmatic veffel, in which is an orifice whofe area is O; the time in which a body falls through A, the altitude of the fluid, is equal to $\sqrt{\frac{A}{l}}$ in feconds; therefore $\frac{B}{O} \times \sqrt{\frac{A}{l}}$ is the time required in feconds.

Let A the altitude of a veffel filled with mercury be 9 inches, l 193 inches, the diameter of the cylindrical veffel 1 inch, and the diameter of the circular aperture $\frac{1}{10}$ of an inch. The time of the difcharge by theory, according to the foregoing formula, will be 86,4 feconds; but by experiment it is found to be 140 feconds nearly; therefore the velocity of the efflux by theory is to be diminifhed

diminiſhed in the ratio of 140 to 86,4; that is, in a ratio between the ratios of the ſquare root of 2 and the ſquare root of 3 to 1.

But this method alſo is ſubject to inaccuracy; for the motion of the fluid is found not to be very regular towards the end of the flux; it will therefore be better to calculate the time in which the veſſel ſhould empty itſelf to a certain depth, which is done in the following manner: The times in which the whole and part would be evacuated are reſpectively $\frac{B}{O} \times \sqrt{\frac{A}{l}}$, and $\frac{B}{O} \times \sqrt{\frac{P}{l}}$, P denoting the height of the part. Therefore the difference is $\frac{B}{O\sqrt{l}} \times \overline{\sqrt{A} - \sqrt{P}}$, in ſeconds.

Thus let the altitude of water in a veſſel wholly and in part filled with water be 16 and 12 inches, the diameter of the cylindric veſſel 5,74, and the diameter of the circular aperture ,2. Then, by the foregoing formula, the time in which the water ſhould ſubſide, according to theory, from the height of 16 to 12 inches, would be 33 ſeconds. But the time actually found by experiment is 53 ſeconds; therefore the velocity determined by theory is to be diminiſhed in the ratio of 53 to 33, or 1,6 to 1, *i. e.* very nearly in the ſame ratio as determined by a former experiment, in which the ſpouting fluid was mercury.